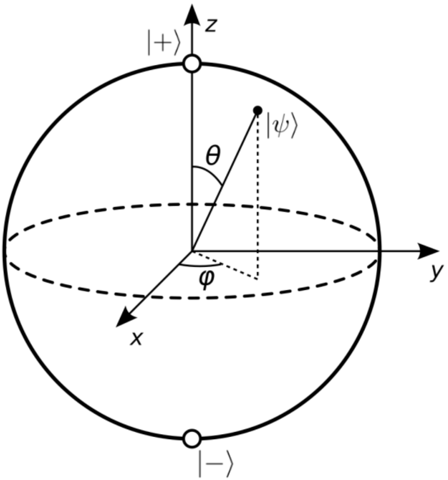
Hopf on the Bloch: From Fiber Bundles to Qubits

Wayne D. Pooley

Outline

1. What is the Bloch Sphere and why do we care?
   - Qubits and Spin
   - Projective space and physical states
2. The Topology Behind the Scenes
   - Normalizing $\mathbb{C}$: the 3-sphere
   - Global phase $\to$ quotienting by $U(1)$
   - Fiber bundles and linked circles
3. The Hopf Fibration
   - Visualizing the linked fibers
   - The map $S^3 \to S^2$
   - Why $S^3 \ncong S^2 \times S^1$

What is the Bloch Sphere?



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Bit of history

Spin Story: Stern-Gerlach Experiment

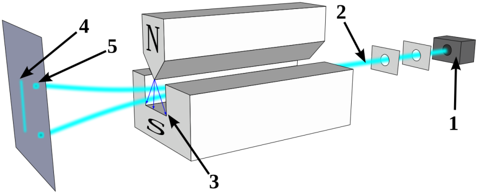

Figure: SG Experiment Apparatus

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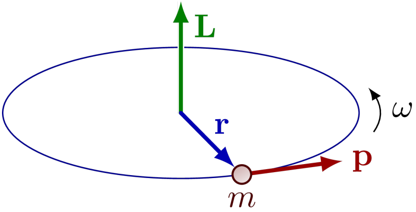
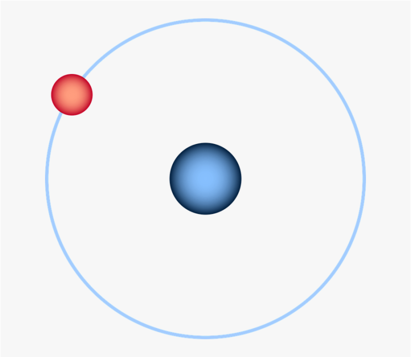
Spin Interlude (*Spin-terlude?*)





State Space

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$|+\rangle$ and $|-\rangle$ scalable by complex coeficients α and β in $\mathbb{C}$.
Thus our state space is $\mathbb{C}^2$ and $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2$.

Normalizing $\mathbb{C} \to S^3$

Motivation from probability theory, we want our vectors $|\psi\rangle$ in the state space $\mathbb{C}^2$:

$$|\psi\rangle \mapsto \frac{|\psi\rangle}{\| \, |\psi\rangle \, \|}$$

The Topology Behind the Scenes

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Effectively, taking the us from 4 real-dimensions ($\mathbb{C}^2 \cong \mathbb{R}^4$ as vector spaces) to 3-real dimensional surface, resulting in the hypersphere S^3. Multiple states are mapped to the same normalized state, called a *normalized spinor*.

We are now in S^3! But... we have some coordinate-redundancy

Global phase equivalence

Multiple normalized spinors correspond to the same physical state under the Born rule.
Example: For two states $|\psi\rangle, |\varphi\rangle$ defined to be $|\psi\rangle = e^{i\theta}|\varphi\rangle$. We have

$$\| \langle\psi|\psi\rangle \|^2 = \| \langle\varphi| e^{-i\theta} \cdot e^{i\theta} |\varphi\rangle \|^2 = \| \langle\varphi|\varphi\rangle \|^2$$

Global phase $\rightarrow$ quotienting by $U(1)$

We aim to set up an equivalence between states that differ by a phase factor:

$$|\psi\rangle \sim e^{i\theta} |\psi\rangle$$

To do this, we can quotient out by all elements $e^{i\theta}$.

Definition: The Unitary Group of dimension 1 ($U(1)$)

$$U(1) := \{z \in \mathbb{C} : |z| = 1\}$$

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Projective Line Crash Course

Train tracks meeting.

Projective Line Crash Course

Definition: Projective Space

Set of equivalence classes under the equivalence relation $\sim$ defined by $x \sim y$ if $x = \lambda y$ for some λ in the field.

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- Wayne Pooley

Definition: Topological Space

Let X be a set, we define a topology on this set τ_X to be the collection of open sets contained in X. Together, these form the topological space (X, τ_X).

Fiber bundles and linked circles

Definition: Fiber Bundles

Let E, B, and F be topological spaces.

- E is our *total space*.
- B is our *base space*.
- F is called the *fiber*.
- A map π from E to B is called the projection of the fiber.

Altogether, these form a *fiber bundle*: (E, B, π, F).

Fiber bundles and linked circles

We shall see that the desired fiber bundle will be

$$(S^3, S^2, \pi, S^1)$$

where $\pi : S^3 \to S^2$.

Fiber bundles and linked circles

There are two main valid ways to get from S^3 to S^2:
The direct way π, is called the *Hopf fibration*.

Property: Local Triviality

Open sets in E are homeomorphic to open sets in the product space B.

$$\varphi : V \to U \times F$$

where V open in E open and U open in B.

Fiber bundles and linked circles

Connecting to S^3 and S^2:

- Product space is $S^2 \times S^1$.
- U open in S^2 and $U \times S^1$ open in $S^2 \times S^1$.

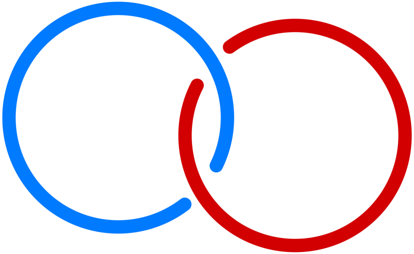
These form the following commutative diagram:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ U & \end{tikzcd}$$

One of the main differences is what the fibers/circles are doing.

Fiber bundles and linked circles

The circles of the S^3 are mutually linked. The simplest link that between two circles is called the Hopf link.



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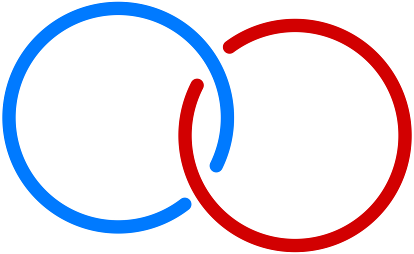

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Why $S^3 \not\cong S^2 \times S^1$

The circles of the $S^2 \times S^1$ are not linked and do not intersect. Thus the topology is trivially projects to the 2-sphere via projection to the first factor.

Visualizing the linked fibers

The interactive Hopf map visualizer by Nico Belmonte (@philogb)
https://philogb.github.io/page/hopf/

Different topologies

Image source: Breaking Bad

The map $S^3 \to S^2$

To quotient S^3 by the $U(1)$, we need to identify it with a group:

Definition: The Special Unitary Group of dimension 2 ($SU(2)$)

$$SU(2) := \{U \in \mathrm{Mat}_{2\times 2} : U^\dagger U = 1, \det U = 1\}$$

The elements take the form of $U = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$.

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To see the connection between S^3 and $SU(2)$ clearly, we establish a bijection:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$$

The map $S^3 \to S^2$

We identify $SU(2)$ with S^3 and it can inherit its topology as a smooth manifold.
Thus our quotient $SU(2)/U(1)$ removes our fibers.

The map $S^3 \to S^2$

But how do we know that this quotient group can be identified with S^2?

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We bridge these concepts with the complex projective line $\mathbb{C}P^1$.

The map $S^3 \to S^2$

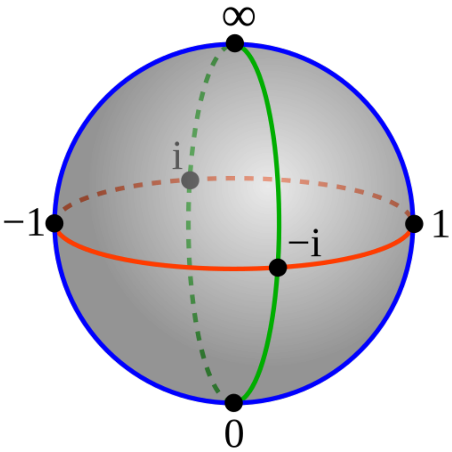

Figure: Riemann Sphere

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The map $S^3 \to S^2$

Quotienting by $U(1)$ sends matrix in $SU(2)$ through the chain

$$\begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \mapsto [\alpha, \beta] \mapsto \begin{cases} \left[\frac{\alpha}{\beta} : 1\right] \mapsto \frac{\alpha}{\beta} & \text{if } \beta \neq 0 \\ [1 : 0] \mapsto \infty & \text{otherwise} \end{cases}$$

Which gets mapped to equivalence classes in $\mathbb{C}P^1$ (i.e. a physical state) and a point on the Riemann sphere.

The map $S^3 \to S^2$

The Riemann sphere has the topology of the 2-sphere- which was our goal! We have reached the Bloch sphere!

Interpretation and Physical Insight

Recall what this did was remove the fibers of our S^3 giving us the *physical states*, which visualize nicely as S^2 or the Bloch sphere.

Explicit Hopf Fibration

Recall:

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : \ |z_1|^2 + |z_2|^2\}$$
$$S^2 = \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^2 = 1\}$$

Explicit Hopf Fibration

Define map $\pi : S^3 \to S^2$ to be

$$\pi(z_1, z_2) = (2\mathrm{Re}(z_1\bar{z_2}), 2\mathrm{Im}(z_1\bar{z_2}), |z_1|^2 - |z_2|^2)$$

Surprise